TRINITY COLLEGE

Year 11 Mathematics Specialist
Test 6 2020

Proof and Complex Numbers

STUDENT'S NAME Solutions - J.C.

DATE: Wednesday 9th September **TIME:** 50 minutes **MARKS:** 52

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Scientific Calculator only, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

1. (2 marks)

Express the following recurring decimal as a fraction. It is not necessary to simplify the fraction.

$0.0\overline{13}$

$x = 0.0\overline{13}\ldots$

$10x = 0.\overline{13}\ldots$

$1000x = 13.\overline{13}\,\overline{13}\ldots$ ✓

$990x = 13$

$x = \frac{13}{990}$ ✓

2. (4 marks)

Prove by contradiction $\sqrt{3}$ is irrational.

assume $\sqrt{3} = \frac{a}{b}$ where a and b are in simplified form.

$3 = \frac{a^2}{b^2}$

$b^2 = \frac{a^2}{3}$ ✓ $\therefore$ a^2 is divisible by 3.

$\therefore$ a is divisible by 3 ✓

let $a = 3m$. (multiple of 3)

$a^2 = 3b^2$

$9m^2 = 3b^2$

$3m^2 = b^2$ ✓ $\therefore$ b^2 is a multiple of 3

$\therefore$ b is a multiple of 3.

a and b are both multiples of 3.

$\therefore$ by contradiction

$\sqrt{3}$ is irrational ✓

3. (4 marks)

Prove the sum of five consecutive odd numbers is a multiple of five.

2n is even

2n + 1 is odd. ✓

$\therefore$ $(2n-3) + (2n-1) + (2n+1) + (2n+3) + (2n+5)$ ✓

$= 10n + 5$

$= 5(2n+1)$ ✓

$\therefore$ is a factor of 5 / multiple of 5 ✓

$\boxed{\text{* Must show odd numbers.}}$

4. (12 marks)

Given $z = 4 + 3i$ and $w = 2 - 5i$ determine:

(a) $w^2 = (2-5i)(2-5i)$ [2]

$= 4 - 20i + 25i^2$

$= -21 - 20i$

(b) $z\overline{w} = (4+3i)(2+5i)$ [2]

$= 8 + 6i + 20i + 15i^2$

$= -7 + 26i$

(c) $\dfrac{w}{z} = \dfrac{2-5i}{4+3i} \times \dfrac{4-3i}{4-3i} = \dfrac{8-20i-6i+15i^2}{16-9i^2}$ [3]

$= \dfrac{-7}{25} - \dfrac{26i}{25}$

(d) $3z - 4w$ $\quad 3(4+3i) - 4(2-5i)$ [2]

$= 12 + 9i - 8 + 20i$

$= 4 + 29i$

(e) $Im\left(\dfrac{1}{z}\right)$ $\quad \dfrac{1}{z} = \dfrac{1}{4+3i} \times \dfrac{4-3i}{4-3i}$ [3]

$\dfrac{1}{z} = \dfrac{4-3i}{16-9i^2}$

$\dfrac{1}{z} = \dfrac{4-3i}{25}$

$Im\left(\dfrac{1}{z}\right) = \dfrac{-3}{25}$

5. (4 marks)

A quadratic equation in the form $x^2 + bx + c = 0$ has one of its roots $7 - 3i$. Determine b and c.

$x = 7 - 3i$

$x = 7 + 3i$

$((x-7)-3i)((x-7)+3i)$ ✓

$(x-7)^2 - (3i)^2$ ✓

$x^2 - 14x + 49 + 9$

$x^2 - 14x + 58$ ✓

$b = -14$ ✓

$c = 58$

6. (6 marks)

Prove

(a) $n^3 - n$ is a multiple of 6, for $n \geq 2$ [3]

$n(n^2 - 1)$

$n(n-1)(n+1)$ ✓

∴ - n or $n+1$ is even
so divisible by 2 ✓

- 3 consecutive numbers
so one is divisible by 3.

given divisible by 2 and 3. ✓
divisible by 6.

(b) $\overline{wz} = \overline{w}\,\overline{z}$ given w and z are complex numbers [3]

let $w = a + bi$

let $z = c + di$ ✓

$wz = ac + adi + bci + bdi^2$

$= [ac - bd] + [ad + bc]i$

LHS $\overline{wz} = [ac - bd] - [ad + bc]i$ ✓

RHS $\overline{w}\,\overline{z} = (a - bi)(c - di)$

$= ac - bci - adi + bdi^2$

$= [ac - bd] - [ad + bc]i$ ✓

$=$ LHS.

7. (8 marks)

(a) Solve $x^2 - 10x + 29 = 0$ [4]

$(x-5)^2 + 4 = 0$ ✓

$(x-5)^2 = -4$ ✓

$x - 5 = \sqrt{4.(-1)}$

$x - 5 = \pm 2i$ ✓

$x = 5 \pm 2i$ ✓

(b) Determine the complex number z given $z - 2\bar{z} = 5 + 6i$ [4]

let $z = a + bi$

$(a + bi) - 2(a - bi) = 5 + 6i$ ✓

$-a + 3bi = 5 + 6i$

$a = -5$ ✓

$3b = 6$

$b = 2$ ✓

$\therefore\ z = -5 + 2i$ ✓

8. (6 marks)

Prove the following conjecture using mathematical induction,

for all $n \geq 1$, $\dfrac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \ldots + x^n$ where $x \neq 1$

• case 1 $n = 1$

$$LHS = \frac{x^2 - 1}{x - 1} \qquad RHS = 1 + x^1$$

$$= \frac{(x+1)(x-1)}{x-1} \qquad = LHS$$

$$= x + 1 \quad ✓ \qquad \therefore \text{ true for } n = 1$$

• assume true for $n = k$

$$\therefore \frac{x^{k+1} - 1}{x - 1} = 1 + x + x^2 + \ldots x^k \quad ✓$$

• case $n = k+1$ $\dfrac{x^{k+2} - 1}{x - 1} = 1 + x + x^2 + \ldots x^k + x^{k+1}$

$$RHS = \underbrace{1 + x + x^2 + \ldots x^k} + x^{k+1} \quad ✓$$

$$= \frac{x^{k+1} - 1}{x - 1} + x^{k+1} \quad ✓$$

$$= \frac{x^{k+1} - 1 + x^{k+1}(x-1)}{x - 1}$$

$$= \frac{x^{k+1} - 1 + x^{k+2} - x^{k+1}}{x - 1} \quad ✓$$

$$= \frac{x^{k+2} - 1}{x - 1}$$

$$= LHS$$

• given true for $n = k+1$ when assumed true for $n = k$ and true for $n = 1$, therefore true for $n = 2$, and then true for $n = 3$ ✓ therefore by induction.

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \ldots x^n \quad \text{where } x \neq 1 \text{ and } n \geq 1$$

9. (6 marks)

Use mathematical induction to prove the following conjecture.

$$2^{n+1}\sin x \cos x \cos(2x)\cos(4x)\ldots\cos(2^n x) = \sin(2^{n+1}x) \text{ for } n \geq 0,\ n \in \mathbb{Z}$$

Case

- let $n = 0$. $LHS = 2^1 \sin x \cos x$

$RHS = \sin 2^1 x$

$= 2 \sin x \cos x$ ✓

- assume true $n = K$

$\therefore 2^{K+1} \sin x \cos x \cos(2x)\cos(4x)\ldots\cos(2^K x) = \sin(2^{K+1}x)$ ✓

- let $n = K+1$

to prove. $2^{K+1+1} \sin x \cos x \cos(2x)\cos(4x)\ldots\cos(2^K x).\cos(2^{K+1}x) = \sin(2^{K+1+1}x)$

$LHS = 2.\underbrace{2^{K+1} \sin x \cos x \cos(2x)\cos(4x)\ldots\cos(2^K x)}\cos(2^{K+1}x)$ ✓

$= 2.\sin(2^{K+1}x).\cos(2^{K+1}x)$ ✓

$RHS = \sin(2^{K+1+1}x)$

$= \sin(2.(2^{K+1}x))$ ✓

$= 2\sin(2^{K+1}x)\cos(2^{K+1}x)$

$= LHS$

$\therefore$ true for $n = K+1$ when assumed true for $n = K$, ✓
given true for $n = 0$, therefore true for $n = 1$, $n = 2$, etc.

$\therefore 2^{n+1} \sin x \cos x \cos(2x)\cos(4x)\ldots\cos(2^n x) = \sin(2^{n+1}x)$ for $n \geq 0$ $n \in \mathbb{Z}$